ON A NEW CURVE FOR THE TRISECTION OF AN ANGLE.

BY DR. WILLIAM HILLHOUSE.

THE following method for the trisection of an angle from 0 to 360°, is believed to be new. The eq'n to the curve has not been found in any book relating to curves that has come to the notice of the writer, nor has its mechanical description ever been seen. It is now offered to those interested in such matters, as a new contribution to the subject, and if it should excite any interest in them, or in any way prove of value in geometry, the object of the writer will be attained.

ABCD, Fig. 1, represents a jointed parallelogram (*AC*, *FE* and *BD* are supposed to be infinite). Into *CD* is fixed firmly the piece *FE*, whose edge *FE* is at right angles to *CD*. This piece *FE* is about one-third the thickness of *CD* to admit of its passing freely under the bar *AC*, which is elevated on the under side for that purpose. The point *F* is also equidistant from the inner edges of *AC* and *BD*. If the bar *BD* be fixed on a plane, and the bar *AC* be depressed, as indicated by the dotted lines in the figure, *AC* will always remain parallel to *BD*, and the bar *FE* will incline to the left, as *fe*, and intersect the edge of *AC*. The p't of intersection of the inner edges of *AC* and *FE* will trace out (by applying a pencil at the point) a locus, represented in part by the line *bpc*, which will be of the fourth order, having four infinite branches, *AH*, *AG*, *BN*, *BK* (Fig. 2). The inner edge of *AC*, when it is parallel to *EF* will be an asymptote to the curve. The eq'n of the curve may be found as follows, by refer'g to Fig. 2.

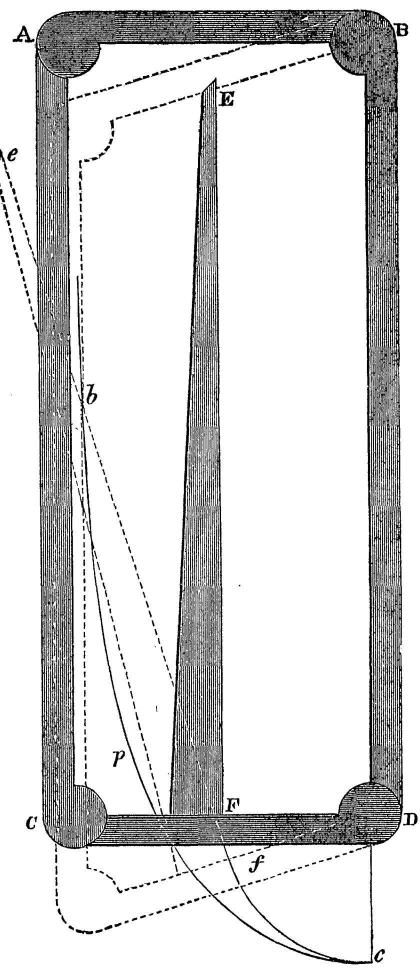

Fig 1.

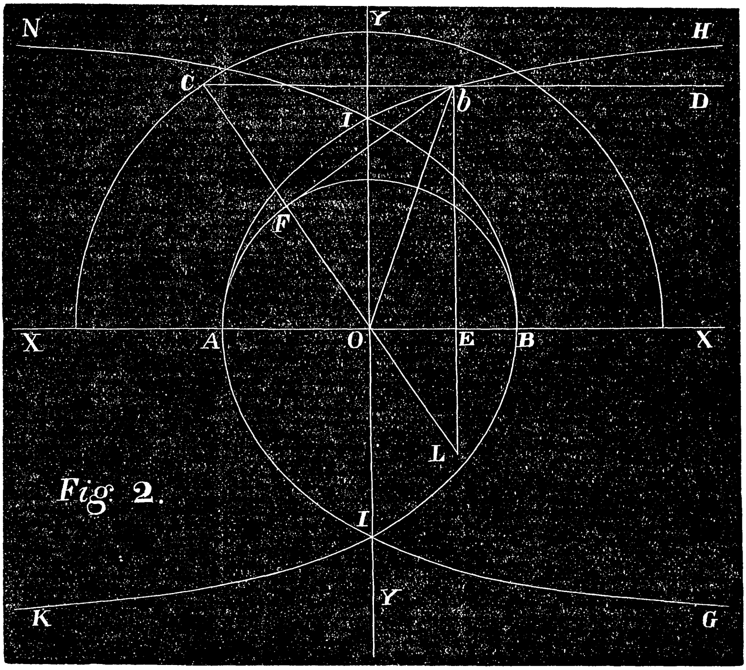

Fig. 2.

Let XX, YY be the rectangular axes; O the origin. With O as centre and radius $OA = a$, describe a semi-circle AFB, and with radius $OC = 2a$ describe a semi-circle XCX. Let b be a point on the curve; through b draw CD parallel to XX, cutting the outer circle in C; from b demit bE perpendicular to XX, and produce it towards L; join CO and produce the line to cut bE in L; now Cb and Ob are constantly equal, and CD is parallel to XX. Then, since the triangles LCb and LOE are similar,

$$Cb : CL :: OE : OL,$$
$$Cb : bL :: OE : EL;$$

or putting $OE = x$, $bE = y$, $OF = OA = a$, $OC = 2a$, $Cb = Ob = r$, $OL = z$, $EL = v$, $CL = 2a+z$, it becomes

$$r : 2a+z :: x : z,$$
$$r : y+v :: x : v;$$

$$\therefore rz = x(2a+z), \qquad z = \frac{2ax}{r-x},$$

$$rv = x(y+v), \qquad v = \frac{xy}{r-x}.$$

But we have $\quad z^2 = x^2 + v^2, \qquad r^2 = x^2 + y^2;$

$$\therefore z^2 = \frac{4a^2x^2}{(r-x)^2} = x^2 + \frac{x^2y^2}{(r-x)^2},$$

$$\text{or } 4a^2 = (r-x)^2 + y^2. \qquad (1)$$

Substituting for r in (1), and reducing, we have finally

$$x^2 = \frac{(2a^2-y^2)^2}{4a^2-y^2}$$

for the equation of the locus.

To trisect any given angle, draw a straight line (Fig. 3) BOX and a straight line bBD at right angles to it. From B lay off $BO = a$; with O as centre and distance BO, describe a circle fBG, and also from the same point a circle with radius $OC = 2a$. Let BOX be the initial line from which the given angle is measured, passing around to the left; let HOX be the angle to be trisected. From O draw OKN at right angles to HO, and place the parallelogram so that the inner edge of CD (Fig. 1) is on the line HO, and the inner edge of BD (Fig. 1) on OKN; depress the side AC of the parallelogram so that the intersection of the lines AC and FE shall be on the line DBb, say the point b; join bO, and from the point b draw bf tangent to the circle BGf; from O through f draw OfC, which will be perpendicular to bf, and join bC. Now the angle $OKB = CbB = HOX$,

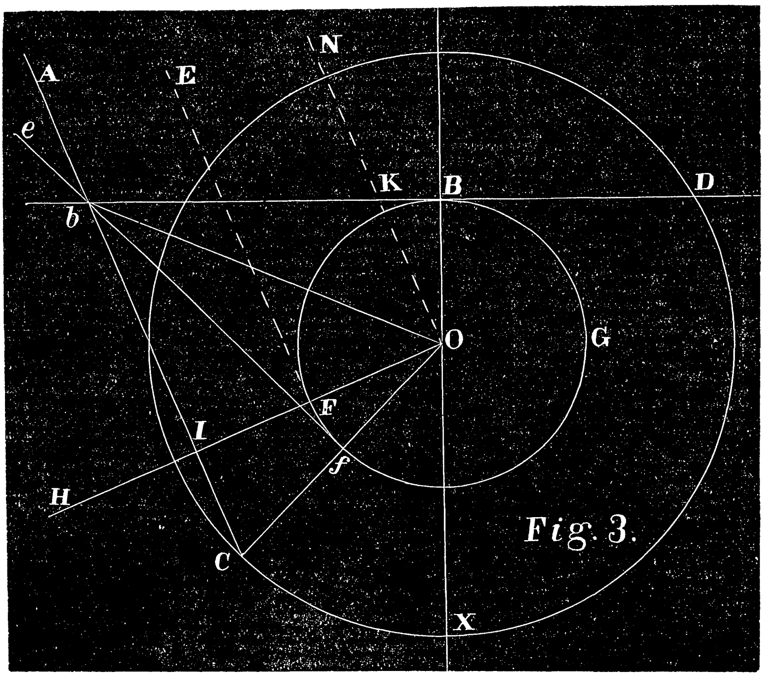

Fig. 3.

and in the right angled triangles bBO, bfO, the side BO = the side Of, since they are radii of the same circle, and the side bO common, therefore the other angles are equal, that is $ObB = Obf$; also in the right triangles bfO, bfC, the side Of = the side fC by construction and the side bf is common, therefore the angle Obf = the angle Cbf. But Obf was shown to be $= ObB$; therefore $Cbf = ObB$, that is, the three angles are equal to each other. But the three angles Cbf, fbO, ObB, make up the angle $CbB = HOX$ the given angle.

Now in the right angled triangles bfC and OIC the angle bCO is common to the two triangles, therefore the remaining angles are equal, that is, the angle COI = the angle Cbf; but Cbf has been shown to be equal to ObB, therefore $COI = ObB$; but ObB is one-third of CbB or HOX, therefore HOC = one-third HOX.

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New Haven, Conn., Jan., 1878.

SOLUTION OF PROB. 405 BY PROF. C. A. VAN VELZER.—To fix the idea take the determinant of the fourth order

$$\begin{vmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \\ a_4 & b_4 & c_4 & d_4 \end{vmatrix} = 0.$$

To the second column add l_1 times the first, m_1 times the third, and n_1 times the fourth, where l_1, m_1, n_1 are chosen to satisfy the equations

$$(l_1-1)a_1+m_1c_1+n_1d_1+b_1 = 0,$$
$$(l_1-1)a_2+m_1c_2+n_1d_2+b_2 = 0,$$
$$(l_1-1)a_3+m_1c_3+n_1d_3+b_3 = 0.$$

These three equations are sufficient to determine the values of (l_1-1), m_1, n_1, but if to these we add a fourth

$$(l_1-1)a_4+m_1c_4+n_1d_4+b_4 = 0$$

these *four* equations form a consistent set in (l_1-1), m_1, n_1, since the determinant of the coefficients (viz. the original determinant) vanishes.

We see that by this first transformation the determinant reduces to

$$\begin{vmatrix} a_1 & a_1 & c_1 & d_1 \\ a_2 & a_2 & c_2 & d_2 \\ a_3 & a_3 & c_3 & d_3 \\ a_4 & a_4 & c_4 & d_4 \end{vmatrix}$$

Now to the second row of this determinant add l_2 times the first, m_2 times the third and n_2 times the fourth, where l_2, m_2, n_2 satisfy the equ'ns